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ALMOST SURE INVARIANCE PRINCIPLES FOR SUMS OF B-VALUED RANDOM VARIABLES WITH APPLICATIONS TO RANDOM FOURIER SERIES AND THE EMPIRICAL CHARACTERISTIC PROCESS

BY

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ABSTRACT. We establish an almost sure approximation of the partial sums of independent, identically distributed random variables with values in a separable Banach space B by a suitable B-valued Brownian motion under the hypothesis that the partial sums can be L^1-closely approximated by finite-dimensional random variables. We show that this hypothesis is satisfied if the given random variables are random Fourier series or related stochastic processes. As an application we obtain an almost sure approximation of the empirical characteristic process by a suitable $\mathbf{C}(K)$-valued Brownian motion whenever the empirical characteristic process satisfies the central limit theorem.

1. Introduction. Let $\{x_j, j \geqslant 1\}$ be a sequence of independent identically distributed random variables with values in a real separable Banach space $(B, \|\cdot\|)$ with mean zero and finite second moment. Let S_n denote the nth partial sum of the sequence. We say that $\{x_j, j \geqslant 1\}$ satisfies the central limit theorem (or that $x = x_1$ satisfies the central limit theorem) if $n^{-1/2}S_n$ converges weakly to a Gaussian measure on B.

In **[17]** it is shown that if $\{x_j, j \geqslant 1\}$ satisfies the central limit theorem then an almost sure invariance principle holds. That is, without changing the probability law of $\{x_j, j \geqslant 1\}$ we can redefine the sequence $\{x_j, j \geqslant 1\}$ on a new probability space on which there exists a B-valued Brownian motion $\{X(t), t \geqslant 0\}$ with covariance

$$Ef(X(1))g(X(1)) = E(f(x_1)g(x_1)), \qquad f, g \in B^*, \tag{1.1}$$

such that

$$\Big\|\sum_{j<t} x_j - X(t)\Big\| = o\big((t \log \log t)^{1/2}\big) \quad \text{a.s.} \tag{1.2}$$

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Also in **[18]**, under the same hypothesis, an L^2-invariance principle is obtained, replacing (1.2) by

$$n^{-1/2} \max_{k<n} \left\| \sum_{j<k} x_j - X(k) \right\| \to 0 \quad \text{in } L^2. \tag{1.3}$$

Let $\{P_s,\ 0 < s \leqslant 1\}$ be a set of uniformly bounded linear operators $P_s\colon B \to B$ with finite-dimensional range such that

$$\sup_{n \geqslant 1} n^{-1/2} E \|S_n - P_s S_n\| \leqslant u(s) \tag{1.4}$$

where $\lim_{s\to 0} u(s) = 0$. It is easy to see and it follows also from a well-known theorem of Hoffmann-Jorgensen [7] and Pisier **[19]** that (1.4) implies the central limit theorem for $\{x_j,\ j \geqslant 1\}$. Therefore, by the above remarks, (1.4) implies an almost sure and an L^2-invariance principle, (1.2) and (1.3), as well.

One of our main objectives in this paper is to obtain an estimate of the size of the "little o" term in (1.2) in terms of the function u in (1.4).

THEOREM 1.1. *Let $\{x_j, j \geqslant 1\}$ be a sequence of independent, identically distributed B-valued random variables, centered at expectations and with finite moments of order $2 + \delta$, where $\delta > 0$. Let $\{P_s,\ 0 < s \leqslant 1\}$ be a set of uniformly bounded linear operators $P_s\colon B \to B$ satisfying* (1.4). *We assume* (*without serious loss of generality*) *that $u(s)$ as given in* (1.4) *is continuous and strictly decreasing as $s \downarrow 0$. Write*

$$u_1(s) = u^{\varepsilon}(s) \quad \textit{where } \varepsilon = \delta/(3 + 3\delta). \tag{1.5}$$

Then the following three results are obtained:

(i) *Let* $\dim P_s B \leqslant \exp(1/s)$ *and $u_1(s)$ satisfy*

$$\frac{u_1(s)}{s} \leqslant \frac{u_1(t)}{t}, \qquad 0 < t \leqslant s \leqslant 1, \tag{1.6}$$

and $u_1(1) = 1$. Then without changing its probability law we can redefine the sequence $\{x_j,\ j \geqslant 1\}$ on a new probability space on which there exists a Brownian motion $\{X(t),\ t \geqslant 0\}$ with the same covariance structure (1.1) *as x_1 such that with probability* 1

$$\left\| \sum_{j<t} x_j - X(t) \right\| \ll g(t, \delta)(t \log\log t)^{1/2} \tag{1.7}$$

where

$$g(t, \delta) = (u(\alpha/\log t))^{\varepsilon/2} \quad \textit{and} \quad \alpha = 13/\min(\delta, 1). \tag{1.8}$$

(ii) *Let* $\dim P_s B \leqslant \exp(1/s)$ *and* $u(s) = s^{\beta}$, $0 < \beta < \infty$. *Then the conclusion of* (i) *holds with* (1.8) *replaced by*

$$g(t, \delta) = (\log t)^{-\beta\varepsilon/2}. \tag{1.9}$$

(iii) *Let* $\dim P_s B \leqslant 1/s$ *and* $u(s) \leqslant s^{\beta}$, $0 < \beta \leqslant 1$. *Then the conclusion of* (i) *holds with* (1.8) *replaced by*

$$g(t, \delta) = t^{-\lambda} \tag{1.10}$$

for some $\lambda > 0$, depending on δ and β only. (*Note that $f(t) \ll g(t)$ means the same as $f(t) = O(g(t))$ as $t \to \infty$.*)

A result related to Theorem 1.1 is Theorem 2 of **[13]**.

A few remarks may be useful in explaining Theorem 1.1. In (i) we show that whenever we can find a $u(s)$ satisfying (1.4) we can make some statement about the "little o" term in (1.2) because (1.6) is satisfied by concave functions. Given any $u(s)$ such that $\lim_{s\to 0} u(s) = 0$ we can find a concave function $w_1(s)$ satisfying $u_1(s) \leqslant w_1(s)$, $0 < s \leqslant 1$ and $\lim_{s\to 0} w_1(s) = 0$. Therefore, we can use w_1 in (i) in place of u_1 and obtain $\ll (w_1(\alpha/\log t))^{1/2}$ as in (1.8) as an improvement over (1.2). Also note that $g(t, \delta) \geqslant (\alpha/\log t)^{\varepsilon/2}$ in (1.8) because of (1.6). To get smaller bounds we can use (ii) or (iii) when they apply. It would be desirable to get a single expression incorporating all these results and which would be valid for all functions u, but we have not been able to do this. Finally, note that the size of $u(s)$ depends on dim P_s. Thus $u(s)$ can be much smaller in (iii) than in (ii). This will be clearer in the examples which follow.

Our main application of Theorem 1.1 will be to the Banach space $\mathbf{C}(K, \tau)$ of continuous complex-valued functions on the metric or pseudometric space (K, τ). Let $N_\tau(K, s)$ denote the minimal number of open balls of radius s in the τ-metric or pseudometric with centers in K that cover K. Then one can find bounded linear operators P_s of dimension $N_\tau(K, s)$ (see Lemma 4.1) such that

$$E\|S_n - P_sS_n\| \leqslant E\Big[\sup_{\tau(v,v')<s;\ v,v'\in K} |S_n(v) - S_n(v')|\Big]. \tag{1.11}$$

Therefore, the term on the right side of (1.11) can be used to obtain $u(s)$ in (1.4). In general, it is not easy to obtain bounds for such expressions. However following, or in some cases slightly modifying, the proofs of some recent results on the central limit theorem in $\mathbf{C}(K, \tau)$ (**[8]**, **[14]**, **[15]**, **[16]**, **[6]**), we can obtain such bounds for certain types of stochastic processes. In Theorem 4.2 this is done for four different classes of processes.

To illustrate our results we will present here the one dealing with the empirical characteristic process. It was S. Csörgö's **[3]** work on almost sure approximation theorems for certain examples of the empirical characteristic process that provided the motivation for this paper. In the following introduction to the empirical characteristic process we follow [3]. Let X be a random variable with values in $\mathbf{R}^N$, with distribution function $F(z)$, $z \in \mathbf{R}^N$ and characteristic function

$$c(v) = \int_{\mathbf{R}^N} e^{i\langle z,v\rangle}\, dF(z) = Ee^{i\langle X,v\rangle}. \tag{1.12}$$

Consider also for $v \in \mathbf{R}^N$

$$\sigma^2(v) = 2(1 - \operatorname{Re} c(v)) = 4\int_{\mathbf{R}^N} \sin^2\tfrac{1}{2}\langle z, v\rangle\, dF(z). \tag{1.13}$$

Let $\{X_k,\ k \geqslant 1\}$ be a sequence of independent copies of X. The empirical distribution function of X, based on a sample of size n, is

$$F_n(z) = n^{-1}\sum_{j\leqslant n} 1\{X_j \leqslant z\}.$$

The Fourier transform of F_n equals

$$c_n(v) = n^{-1} \sum_{j < n} e^{1\langle X_j, v\rangle} \tag{1.14}$$

and is called the empirical characteristic function of X. The empirical characteristic process is defined as

$$C(v, t) = \sum_{j < t} (e^{i\langle X_j, v\rangle} - c(v)), \qquad v \in \left[-\tfrac{1}{2}, \tfrac{1}{2}\right]^N, t \geqslant 0. \tag{1.15}$$

It is necessary to restrict attention to compact subsets of $\mathbf{R}^N$ (see **[3]** for details).

In view of (1.15) the question of weak convergence of $n^{-1/2}C(\cdot, n)$ is the same as that of the central limit theorem for $e^{i\langle X, v\rangle} - c(v)$. If the sequence $\{n^{-1/2}C(\cdot, n), n \geqslant 1\}$ converges weakly on $\mathbf{C}([-\frac{1}{2}, \frac{1}{2}]^N)$ the limit must be a Gaussian process G, with continuous sample paths and with covariance

$$E\left\{(e^{i\langle X, v\rangle} - c(v))(e^{-i\langle X, v'\rangle} - c(-v'))\right\} = c(v - v') - c(v)c(-v').$$

The spectral representation of this process is

$$G(v) = \int_{\mathbf{R}^N} e^{i\langle z, v\rangle}\, d(b(F(z)) - b(1)c(v)),$$

where b is standard Brownian motion. Therefore in order for $n^{-1/2}C(\cdot, n)$ to have a weak limit in $\mathbf{C}([-\frac{1}{2}, \frac{1}{2}]^N)$ the stationary Gaussian process

$$\int_{\mathbf{R}^N} e^{i\langle z, v\rangle}\, db(F(z)), \qquad v \in \left[-\tfrac{1}{2}, \tfrac{1}{2}\right]^N,$$

must have a version with continuous paths. By the Dudley-Fernique theorem this happens if and only if

$$\int_0^\infty \left(\log N_\sigma\left(\left[-\tfrac{1}{2}, \tfrac{1}{2}\right]^N, \varepsilon\right)\right)^{1/2} d\varepsilon < \infty \tag{1.16}$$

for σ given in (1.13) (see **[15**, relations (3), (4) and (5)**]**). It was shown in **[15]**, in the case $N = 1$, that whenever (1.16) holds $n^{-1/2}C(\cdot, n)$ does converge weakly to a Gaussian limit. The result for $N > 1$ has the same proof as the result for $N = 1$. It is given in Theorem 4.2 and Remark 4.3 along with central limit theorems for many related stochastic processes.

Consequently, if (1.16) holds then in view of (1.2) and (1.3)

$$\sup_{v \in \left[-\frac{1}{2}, \frac{1}{2}\right]^N} |C(v, t) - H(v, t)| = o\left((t \log\log t)^{1/2}\right) \quad \text{a.s.} \tag{1.17}$$

and $o(t^{1/2})$ in L^2, where $\{H(v, t), v \in [-\frac{1}{2}, \frac{1}{2}]^N, t \geqslant 0\}$ is a Gaussian process with mean zero and convariance function

$$E\left(H(v, t)\overline{H(v', t)}\right) = \min(t, t')(c(v - v') - c(v)c(-v')). \tag{1.18}$$

In Theorem 1.2 which is an application of Theorem 1.1 we refine (1.17). Let

$$R(x) = P(|X| \leqslant x), \tag{1.19}$$

$$\hat{\sigma}(s) = 2\left(s^2 \int_0^{1/s} x(1 - R(x))\, dx\right)^{1/2}, \tag{1.20}$$

and

$$\psi(s) = \int_0^{\hat{\sigma}(s)} \left(\log N_\sigma\left(\left[-\tfrac{1}{2}, \tfrac{1}{2}\right]^N, \varepsilon\right)\right)^{1/2} d\varepsilon + \hat{\sigma}(s). \tag{1.21}$$

THEOREM 1.2. *Let $\{X_j, j \geqslant 1\}$ be a sequence of independent identically distributed random variables with values in $\mathbf{R}^N$ and distribution function F. Employing the notation given above, we can redefine the sequence $\{X_j, j \geqslant 1\}$ without changing its probability law, on a new probability space on which there exists a mean zero Gaussian process $\{H(v, t), v \in [-\frac{1}{2}, \frac{1}{2}]^N, t \geqslant 0\}$ with covariance function* (1.18) *such that with probability* 1

$$\sup_{v \in \left[-\frac{1}{2}, \frac{1}{2}\right]^N} |C(v, t) - H(v, t)| \ll w^{1/2}\left(B_N t^{-13N}\right)(t \log\log t)^{1/2}. \tag{1.22}$$

Here $w(s)$ is a concave majorant of $\psi^{1/4}(B_N \exp(-(sN)^{-1}))$, B_N is a constant depending on N only and ψ is given in (1.21). *If*

$$1 - R(x) \ll 1/\log x(\log\log x)^{2g}, \qquad g > 1, \tag{1.23}$$

then the error term in (1.22) *is*

$$\ll (\log\log t)^{-(g-1)/8}(t \log\log t)^{1/2}. \tag{1.24}$$

If

$$1 - R(x) \ll (\log x)^{-g}, \qquad g > 1, \tag{1.25}$$

then the error term in (1.22) *is*

$$\ll (\log t)^{-(g-1)/8}\,(t\log\log t)^{1/2}. \tag{1.26}$$

If

$$1 - R(x) \ll x^{-g}, \qquad g > 0, \tag{1.27}$$

then the error term in (1.22) *can be replaced by a term which is*

$$\ll t^{1/2-\lambda} \tag{1.28}$$

for some $\lambda > 0$, depending only on g and N.

We add a few remarks concerning (1.22). Since (1.16) is necessary and sufficient for the central limit theorem to hold for the sequence $\{e^{i\langle X_j, v\rangle} - c(v);\ j \geqslant 1, v \in [-\frac{1}{2}, \frac{1}{2}]^N\}$ we see from (1.21) and (1.22) that whenever the central limit theorem does hold for this sequence we obtain an estimate for the "little o" term in (1.17). In the rest of Theorem 1.2 using smooth upper bounds for $1 - R(x)$ we can obtain cleaner results. To help interpret all this let us note that if $1 - R(x)$ is convex for x sufficiently large then (1.16) holds if and only if

$$\int_2^\infty \frac{(1 - R(x))^{1/2}}{x(\log x)^{1/2}}\, dx < \infty \tag{1.29}$$

(see Theorem 3.2, Chapter 4, of **[10]**). Thus the bounds in (1.23) and (1.25) are about as large as they possibly can be. Actually, bounds can be found for all slowly varying $1 - R(x)$ satisfying (1.29). The case considered in (1.27) and (1.28) with a sharper estimate of the error term is due to S. Csörgö **[3]** for $N = 1$; for $N > 1$ it was obtained independently by him also with a sharper estimate of the error term. Some other applications of Theorem 1.1 as well as the proof of Theorem 1.2 will be given in §4. The proof of Theorem 1.1 is given in §3. §2 contains the preliminaries. Some further applications of these results are mentioned in §5.

2. Preliminaries. For the proof of Theorem 1.1 we need several theorems which we will quote for the reader's convenience. Recall that the Prohorov distance $\pi(\mu, \nu)$ of two probability measures μ and ν on a separable metric space (S, σ) is defined as

$$\pi(\mu, \nu) = \inf\{\varepsilon > 0: \mu(A) \leqslant \nu(A^{\varepsilon}) + \varepsilon \text{ for all closed subsets } A \text{ of } S\}.$$

Here $A^{\varepsilon} = \{x \in S\colon\ \sigma(x, A) < \varepsilon\}$. The following result is a special case of Dehling's [**4**] recent generalization of a theorem of Yurinskii [**20**].

THEOREM 2.1. *Let $\{\xi_j, j \geqslant 1\}$ be a sequence of independent, identically distributed random variables in a d-dimensional Banach space B, centered at expectations and with finite $(2 + \gamma)$ moments where $0 < \gamma \leqslant 1$. Let $S_n = \sum_{j \leqslant n} \xi_j$, let μ_n denote the distribution of $n^{-1/2}S_n$ and let μ be the mean zero Gaussian measure with covariance function $E\{f(\xi_1)g(\xi_1)\}, f, g \in B^*$. Then*

$$\pi(\mu_n, \mu) \leqslant Cd^{4/3}n^{-\gamma/9}$$

where C only depends on $E|\xi_j|^{2+\gamma}$.

To avoid misunderstandings a few remarks might be in order. Yurinskii's [**20**] Theorem 1 deals with independent random variables assuming values in a finite-dimensional inner product space (i.e. a space where the distance is given by an inner product) and having finite third moments. Dehling's [**4**] Theorem 1 partially generalizes Yurinskii's theorem to weakly stationary absolutely regular sequences of finite-dimensional random variables having uniformly bounded moments of order $2 + \gamma$. Theorem 2.1 follows at once from Dehling's [**4**] Proposition 5.1 and Lemma 5.1. Except for the exponents $4/3$ and $-\gamma/9$ it is also contained in his Theorem 1a.

We also need the following result which is given in a more general setting as Theorem 3 in [**17**].

THEOREM 2.2. *Let $\{B_k, m_k, k \geqslant 1\}$ be a sequence of complete separable metric spaces. Let $\{Z_k, k \geqslant 1\}$ be a sequence of independent random variables with values in B_k. Denote the distribution of Z_k by F_k and let $\{G_k, k \geqslant 1\}$ be a sequence of distributions on B_k such that the Prohorov distance $\pi(F_k, G_k) \leqslant \rho_k$, $\rho_k > 0$. Then without changing its distribution we can redefine the sequence $\{Z_k, k \geqslant 1\}$ on a richer probability space on which there exists a sequence $\{Y_k, k \geqslant 1\}$ of independent random variables Y_k with distribution G_k such that for all $k \geqslant 1$*

$$P\{m_k(Z_k, Y_k) \geqslant 2\rho_k\} \leqslant 2\rho_k.$$

The following theorem is due to Kuelbs [**12**, Lemma 2.1].

THEOREM 2.3. *Let $\{\xi_j,\ j \geqslant 1\}$ be a sequence of independent B-valued random variables, centered at expectations and satisfying $\|\xi_j\| \leqslant cb_n$ $(1 \leqslant j \leqslant n)$ where c and b_n are constants. Let $T_n = \sum_{j \leqslant n} \xi_j$. Then for any ρ with $\rho c \leqslant 1$,*

$$P\{\|T_n\| \geqslant 2\rho b_n\} \leqslant \exp\left\{-\rho^2 + \tfrac{1}{2}\rho^2\left(1 + \tfrac{1}{2}\rho c\right)b_n^{-2}\sum_{j<n} E\|\xi_j\|^2 + \tfrac{1}{2}\rho b_n^{-1}E\|T_n\|\right\}.$$

3. Proof of Theorem 1.1.

3.1. *Case* (i). By (1.6) and the fact that $u_1(1) = 1$ we have

$$u_1(s) \geqslant s. \tag{3.1}$$

Consider the function $(su_1(s))^{-1}$. This function is continuous and strictly decreasing for $0 < s < 1$. We define the strictly decreasing sequence $\{s_k\}$, $\lim_{s\to\infty} s_k = 0$ as the unique solution of the equation

$$k = \alpha(s_k u_1(s_k))^{-1}. \tag{3.2}$$

Note that

$$u_1(s_k) \geqslant (\alpha/k)^{1/2} \tag{3.3}$$

since by (3.1) $u_1(s_k) \geqslant s_k = \alpha/(ku_1(s_k))$. We also define

$$t_k = \exp(\alpha/s_k) \tag{3.4}$$

and

$$n_{k,\gamma} = t_k^\gamma - t_{k-1}^\gamma, \qquad 0 < \gamma \leqslant 1. \tag{3.5}$$

For simplicity we denote $n_{k,1} = n_k$.

There are a number of estimates needed in the proof of Theorem 1.1 which we give in the next lemma.

LEMMA 3.1. *We have*

$$u_1(s_k)/(2\alpha) \leqslant s_k^{-1} - s_{k-1}^{-1} \leqslant u_1(s_k)/\alpha, \tag{3.6}$$

$$t_k^\gamma u_1(s_k) \ll n_{k,\gamma} \ll t_k^\gamma u_1(s_k), \qquad 0 < \gamma \leqslant 1, \tag{3.7}$$

$$\sum_{k<m} n_k^{1/2} u_1^{3/2}(s_k) \ll t_m^{1/2} u_1(s_m), \tag{3.8}$$

$$n_k^{-\tau} t_k^\sigma \ll t_k^{(\sigma-\tau)/2} \ll \exp(-k^{1/4}), \qquad 0 \leqslant \sigma < \tau, \tag{3.9}$$

$$\sum_{k<m} t_k^{1/2-\lambda} \ll t_m^{1/2-\lambda/2}, \qquad 0 < \lambda < \tfrac{1}{2}. \tag{3.10}$$

PROOF. We begin with (3.6). By (3.2)

$$\begin{aligned} \frac{1}{\alpha} &= \frac{1}{s_k u_1(s_k)} - \frac{1}{s_{k-1}u_1(s_{k-1})} \\ &= \frac{(s_{k-1} - s_k)u_1(s_{k-1}) - s_k(u_1(s_k) - u_1(s_{k-1}))}{s_k s_{k-1} u_1(s_{k-1}) u_1(s_k)} \\ &\leqslant (s_k^{-1} - s_{k-1}^{-1})\left((u_1(s_k))^{-1} + (u_1(s_{k-1}))^{-1}\right) \leqslant 2(s_k^{-1} - s_{k-1}^{-1})/u_1(s_k) \end{aligned} \tag{3.11}$$

which gives the left side of (3.6). Note that we have used

$$(3.12)\qquad (u_1(s_{k-1}) - u_1(s_k))/u_1(s_k) \leqslant (s_{k-1} - s_k)/s_k$$

which is an immediate consequence of (1.6). For the right side of (3.6) we simply observe that by the first two lines of (3.11)

$$\frac{1}{\alpha} \geqslant \frac{s_{k-1} - s_k}{s_k s_{k-1} u_1(s_k) u_1(s_{k-1})} = \left(s_k^{-1} - s_{k-1}^{-1}\right)/u_1(s_k).$$

By (3.6) we see that for k sufficiently large $s_k^{-1} - s_{k-1}^{-1}$ is small; thus by (3.4) and (3.5)

$$n_{k,\gamma} = t_k^\gamma - t_{k-1}^\gamma \sim t_k^\gamma \alpha\left(s_k^{-1} - s_{k-1}^{-1}\right)$$

and (3.7) follows from (3.6).

We proceed to obtain (3.8). By (3.5) and (3.7) we have

$$(3.13)\qquad \sum_{k<m} t_k^\gamma u_1(s_k) \ll t_m^\gamma = \sum_{k<m} n_k^\gamma \ll \sum_{k<m} t_k^\gamma u_1(s_k).$$

Since by (3.2)

$$(k+1)/\alpha = (s_{k+1}u_1(s_{k+1}))^{-1} \geqslant (s_k u_1(s_k))^{-1} \cdot s_k/s_{k+1} \geqslant (k/\alpha)(s_k/s_{k+1}),$$

we obtain

$$(3.14)\qquad s_k/s_{k+1} \leqslant 1 + 1/k.$$

Hence by (3.12) and (3.3)

$$u_1(s_k) - u_1(s_{k+1}) \leqslant u_1(s_k)\frac{s_k - s_{k+1}}{s_{k+1}} \leqslant u_1(s_k)/k \leqslant u_1^3(s_k)/\alpha.$$

By summation by parts and (3.13) we thus obtain

$$\begin{aligned}\sum_{k<m} t_k^{1/2} u_1^2(s_k) &= \left(\sum_{k<m} t_k^{1/2} u_1(s_k)\right) u_1(s_m) \\ &\quad + \sum_{k<m}\left(\sum_{j<k} t_j^{1/2} u_1(s_j)\right)(u_1(s_k) - u_1(s_{k+1})) \\ &\ll t_m^{1/2} u_1(s_m) + \sum_{k<m} t_k^{1/2} u_1^3(s_k) \ll t_m^{1/2} u_1(s_m)\end{aligned}$$

since $u_1(s_k) \to 0$. Inequality (3.8) follows now from (3.7) in the case $\gamma = 1$.

Next we note that by (3.2) and (3.3)

$$(\alpha k)^{1/2} \leqslant \alpha/s_k \leqslant k$$

so that

$$(3.15)\qquad e^{(\alpha k)^{1/2}} \leqslant t_k \leqslant e^k.$$

Hence by (3.7) and (3.3) we get (3.9). For (3.10) we simply note that

$$\sum_{k<m} t_m^{1/2-\lambda} \leqslant m t_m^{1/2-\lambda} \ll t_m^{1/2-\lambda/2}, \qquad 0 < \lambda < \tfrac{1}{2}.$$

This completes the proof of Lemma 3.1.

LEMMA 3.2. *As* $k \to \infty$,

$$\Big\| \sum_{t_{k-1} < j \leqslant t_k} \big(x_j - P_{s_k} x_j\big) \Big\| \ll u(s_k)^{3\varepsilon/2} (n_k \log\log t_k)^{1/2} \quad a.s.$$

PROOF. We write

$$E\|x_j - P_s x_j\|^2 = E\big\{ \|x_j - P_s x_j\|^{\delta/(1+\delta)} \|x_j - P_s x_j\|^{(2+\delta)/(1+\delta)} \big\}.$$

Then by Hölder's inequality with $p = (1+\delta)/\delta$ and $q = 1 + \delta$ we get

$$E\|x_j - P_s x_j\|^2 \leqslant \big(E\|x_j - P_s x_j\|\big)^{\delta/(1+\delta)} \big(E\|x_j - P_s x_j\|^{2+\delta}\big)^{1/(1+\delta)}.$$

Let $M = 1 + \sup_{0 < s \leqslant 1} \|P_s\|$. We always can normalize so that

$$M \cdot E\|x_1\|^{2+\delta} \leqslant 1. \tag{3.16}$$

Thus by (1.4) and (1.5)

$$E\|x_j - P_s x_j\|^2 \leqslant u(s)^{3\varepsilon}. \tag{3.17}$$

Put

$$z_j = z_j(s) = x_j - P_s x_j.$$

Before we apply Theorem 2.3 we have to truncate the z_j's and to center the truncated random variables. We put

$$z_j' = z_j 1\big\{ \|x_j\| \leqslant j^{1/(2+\delta)} \big\}.$$

Then by (3.17)

$$E\|z_j'\|^2 \leqslant E\|z_j\|^2 \leqslant u(s)^{3\varepsilon}. \tag{3.18}$$

Since $Ez_j = 0$ we have by (3.16)

$$\begin{aligned} \|Ez_j'\| &= \|E\{z_j 1\{\|x_j\| > j^{1/(2+\delta)}\}\}\| \\ &\leqslant ME\|x_j\|^{2+\delta} j^{-(1+\delta)/(2+\delta)}. \end{aligned}$$

Thus by (3.5), (3.7) and (3.9)

$$\begin{aligned} \sum_{t_{k-1} < j \leqslant t_k} \|Ez_j'\| &\ll t_k^{1/(2+\delta)} - t_{k-1}^{1/(2+\delta)} \ll t_k^{1/(2+\delta)} u(s_k)^{3\varepsilon/2} \\ &\ll n_k^{1/2-\lambda} u(s_k)^{3\varepsilon/2} \end{aligned}$$

for some $\lambda > 0$. Similarly, since $z_j - z_j' = z_j 1\{\|x_j\| > j^{1/(2+\delta)}\}$,

$$E\Big\| \sum_{t_{k-1} < j \leqslant t_k} z_j - z_j' \Big\| \leqslant \sum_{t_{k-1} < j \leqslant t_k} E\|z_j - z_j'\| \ll n_k^{1/2-\lambda} u(s_k)^{3\varepsilon/2}.$$

Thus by (1.4)

$$E\Big\| \sum_{t_{k-1} < j \leqslant t_k} z_j' \Big\| \ll n_k^{1/2} u(s_k)^{3\varepsilon/2}.$$

We choose $b_k = 4n_k^{1/2}u(s_k)^{3\varepsilon/2}$, $c = c_k = (M/2)t_k^{-\delta/(8+4\delta)}u(s_k)^{-3\varepsilon/2}$ and $\rho = 4(\log\log t_k)^{1/2}$ in Theorem 2.3 and obtain for sufficiently large k

$$P\left\{\left\|\sum_{t_{k-1}<j\leqslant t_k} z_j'(s_k)\right\| \geqslant 9\rho n_k^{1/2}u(s_k)^{3\varepsilon/2}\right\}$$

$$\leqslant P\left\{\left\|\sum_{t_{k-1}<j\leqslant t_k} z_j'(s_k) - Ez_j'(s_k)\right\| \geqslant 8\rho n_k^{1/2}u(s_k)^{3\varepsilon/2}\right\}$$

$$\leqslant \exp\left(-\rho^2 + \tfrac{1}{2}\rho^2(1+o(1))/16 + \rho\cdot o(1)\right) \ll \exp\left(-\tfrac{1}{2}\rho^2\right) \ll k^{-2}$$

by (3.3), (3.7), (3.9) and (3.15). Hence by the Borel Cantelli lemma

$$\left\|\sum_{t_{k-1}<j\leqslant t_j} z_j'\right\| \ll (n_k \log\log t_k)^{1/2}u(s_k)^{3\varepsilon/2} \quad \text{a.s.}$$

It remains to show that

$$\sum_{k\geqslant 1} P\left\{z_j' \neq z_j \text{ for some } j \text{ with } t_{k-1} < j \leqslant t\right\} < \infty.$$

Then an application of the Borel Cantelli lemma and the last inequality will give

$$\left\|\sum_{t_{k-1}<j\leqslant t_k} z_j\right\| \ll (n_k \log\log t_k)^{1/2}u(s_k)^{3\varepsilon/2} \quad \text{a.s.,}$$

the conclusion of the lemma. We have

$$\sum_{t_{k-1}<j\leqslant t_k} P\{z_j' \neq z_j\} \leqslant \sum_{t_{k-1}<j\leqslant t_k} P\left\{\|x_j\| > j^{1/(2+\delta)}\right\}$$

$$= \sum_{t_{k-1}<j\leqslant t_k} P\left\{\|x_1\|^{2+\delta} > j\right\}.$$

We sum these last terms over $k \geqslant 1$ and obtain a convergent series since $E\|x_1\|^{2+\delta} < \infty$. This concludes the proof of the lemma.

By the remark following (1.4) the sequence $\{x_j, j \geqslant 1\}$ satisfies the central limit theorem. Hence there exists a Gaussian measure μ with the same covariance as x_1. Let $\{X(t), t \geqslant 0\}$ be a B-valued Brownian motion with $L(X(1)) = \mu$.

LEMMA 3.3. *As $k \to \infty$*

$$\|X(t_k) - X(t_{k-1}) - P_{s_k}(X(t_k) - X(t_{k-1}))\| \ll u(s_k)^{3\varepsilon/2}(n_k \log\log t_k)^{1/2} \quad a.s.$$

PROOF. As we noted above $\{x_j, j \geqslant 1\}$ satisfies the central limit theorem. Hence the measures induced on B by $n^{-1/2}(S_n - P_sS_n)$ converge weakly to $X(1) - P_sX(1)$. Since $\{x \in B: \|x - P_sx\| > \tau\}$ is an open set in B we have by (1.4) and a standard theorem on the weak convergence of probability measures that

$$P\{\|X(1) - P_sX(1)\| > 3u(s)\}$$

$$\leqslant \liminf_{n\to\infty} P\left\{n^{-1/2}\|S_n - P_sS_n\| > 3u(s)\right\} < \frac{1}{3}.$$

Since $X(1) - P_sX(1)$ is a bounded Banach space valued Gaussian random variable we obtain from the Fernique, Landau-Shepp lemma

$$P\{\|X(1) - P_sX(1)\| > \rho u(s)\} \leqslant \exp(-\delta^2/\alpha^2) \quad \text{for some } \alpha < \infty \text{ and all } \rho \geqslant 1.$$

By the nature of Brownian motion this inequality is equivalent to

$$P\left\{ \|X(n) - P_s X(n)\| > \rho n^{1/2} u(s) \right\} \leqslant \exp(-\rho^2/\alpha^2).$$

We put $\rho = 4\alpha(\log\log t_k)^{1/2}$, $s = s_k$ and $n = n_k$ and using the fact that $L(X(n_k)) = L(X(t_k) - X(t_{k-1}))$ and (3.15) we obtain the result via the Borel Cantelli lemma.

LEMMA 3.4. *As* $k \to \infty$

$$\sup_{t_{k-1} < t \leqslant t_k} \left\| \sum_{t_{k-1} < j \leqslant t} x_j \right\| \ll u(s_k)^{\varepsilon/2} (t_k \log\log t_k)^{1/2} \quad a.s.$$

PROOF. Again we truncate the random variables x_j at $j^{1/(2+\delta)}$. Set

$$x_j' = x_j 1\left\{ \|x_j\| < j^{1/(2+\delta)} \right\}, \qquad x_j'' = x_j - x_j',$$

and

$$S_n' = \sum_{j \leqslant n} x_j'.$$

Since $E\|x_1\|^{2+\delta} < \infty$ we have by stationarity

$$\sum_{j \geqslant 1} P(x_j'' \neq 0) = \sum_{j \geqslant 1} P\left(\|x_j\| > j^{1/(2+\delta)}\right) < \infty$$

and thus by the Borel Cantelli lemma

$$\sum_{j \geqslant 1} \|x_j''\| < \infty \quad \text{a.s.} \tag{3.19}$$

Now by (3.7) there exists a constant C such that

$$\begin{aligned} &\max_{t_{k-1} < n \leqslant t_k} P\left\{ \|S_n' - S_{t_{k-1}}'\| \geqslant C u(s_k)^{\varepsilon/2} (t_k \log\log t_k)^{1/2} \right\} \\ &\qquad \leqslant \max_{t_{k-1} < n \leqslant t_k} P\left\{ (n - t_{k-1})^{-1/2} \|S_n' - S_{t_{k-1}}'\| \right. \\ &\qquad\qquad \left. \geqslant 4\left(n_k \log\log t_k / (n - t_{k-1})\right)^{1/2} \right\} \\ &\qquad \leqslant \max_{t_{k-1} < n \leqslant t_k} P\left\{ (n - t_{k-1})^{-1/2} \|S_n' - S_{t_{k-1}}'\| \geqslant 4(\log\log t_k)^{1/2} \right\}. \end{aligned} \tag{3.20}$$

In view of (3.19) $\{x_j', j \geqslant 1\}$ satisfies the central limit theorem since $\{x_j, j \geqslant 1\}$ does. Thus the last term in (3.20) goes to zero as $k \to \infty$. We refer the reader to Lemma 3.21 of **[2]**, p. 45. This lemma is proved for real-valued random variables; however, the extension to B-valued random variables is immediate. Hence we obtain for some constant D

$$\begin{aligned} &P\left\{ \max_{t_{k-1} < t \leqslant t_k} \|S_t' - S_{t_{k-1}}'\| \geqslant 32 u(s_k)^{\varepsilon/2} (D t_k \log\log t_k)^{1/2} \right\} \\ &\qquad \ll P\left\{ \|S_{t_k}' - S_{t_{k-1}}'\| \geqslant 16 u(s_k)^{\varepsilon/2} (D t_k \log\log t_k)^{1/2} \right\} \\ &\qquad \ll P\left\{ \|S_{t_k}' - S_{t_{k-1}}'\| \geqslant 16 (n_k \log\log t_k)^{1/2} \right\} \end{aligned} \tag{3.21}$$

by (3.7). Since $E\|P_1(S_{t_k} - S_{t_{k-1}})\| \ll n_k^{1/2}$ we obtain by (1.4) with $s = 1$,

$$E\|S_{t_k} - S_{t_{k-1}}\| \ll n_k^{1/2}.$$

Now

$$E\|S_{t_k} - S_{t_{k-1}} - (S'_{t_k} - S'_{t_{k-1}})\|$$

$$\leqslant \sum_{t_{k-1}<j\leqslant t_k} E\|x_j''\| = \sum_{t_{k-1}<j\leqslant t_k} E\{\|x_j\| 1(\|x_j\| \geqslant j^{1/(2+\delta)})\}$$

$$\leqslant \sum_{t_{k-1}<j\leqslant t_k} E\|x_1\|^{2+\delta} j^{-(1+\delta)/(2+\delta)} \ll n_k^{1/(2+\delta)}.$$

Hence

$$E\|S'_{t_k} - S'_{t_{k-1}}\| \ll n_k^{1/2}.$$

We now can find an upper bound for the last term in (3.21) using Theorem 2.3 with $b_k = 2n_k^{1/2}$, $c = c_k = n_k^{-\delta/(8+4\delta)}$ and $\rho = 4(\log\log t_k)^{1/2}$. By (3.3), (3.7) and (3.15) it follows that $\rho c \to 0$. Thus we obtain for the probability in (3.21) the bound $\ll k^{-2}$ using (3.15) once more. The lemma follows now from the Borel Cantelli lemma and (3.19).

LEMMA 3.5. *As* $k \to \infty$,

$$\sup_{t_{k-1}\leqslant t\leqslant t_k} \|X(t) - X(t_k)\| \ll u(s_k)^{\varepsilon/2}(t_k \log\log t_k)^{1/2} \quad a.s.$$

PROOF. Since $X(t)$ is a Brownian motion

$$\sup_{t_{k-1}\leqslant t\leqslant t_k} \|X(t) - X(t_k)\| \leqslant \sup_{t_{k-1}<j\leqslant t_k} \|X(j) - X([t_{k-1}] + 1)\| + 2\sup_{0\leqslant t\leqslant 1}\|X(t)\|, \tag{3.22}$$

where j is an integer and $[\cdot]$ denotes integral part. The first term in (3.22) is estimated exactly as in Lemma 3.4 (or by applying Lemma 3.4 to the increments $X(j) - X(j-1)$) and the second term in (3.22) is just a finite random variable.

We have already estimated various quantities involving the limiting Brownian motion. Now we must construct a probability space on which the sequence $\{x_j, j \geqslant 1\}$ and the Brownian motion $\{X(t), t \geqslant 0\}$ are defined in order to give a meaning to (1.7). Consider

$$Z_k = n_k^{-1/2} \sum_{t_{k-1}<j\leqslant t_k} P_{s_k} x_j \tag{3.23}$$

and denote by d_k the dimension of P_{s_k}. In (i) $d_k \leqslant \exp(1/s_k)$. Let F_k denote the distribution of Z_k and let G_k denote the distribution of a mean zero Gaussian vector of dimension d_k and with the same covariance matrix as Z_k. From Theorem 2.1, (3.4) (1.8) and (3.9)

$$\rho_k = \pi(F_k, G_k) \ll n_k^{-\delta/9}\exp(4/3s_k) \ll t_k^{-\lambda_1} \ll \exp(-k^{1/4}) \tag{3.24}$$

for some $\lambda_1 > 0$. Therefore, by Theorem 2.2 there exists without loss of generality, a sequence of independent random vectors Y_k with distribution G_k such that $P\{\|Z_k - Y_k\| \geqslant 2\rho_k\} \leqslant 2\rho_k$. Hence by (3.5), (3.24) and the Borel Cantelli lemma

$$\|n_k^{1/2}Z_k - (t_k - t_{k-1})^{1/2} Y_k\| \ll t_k^{1/2-\lambda_1} \quad \text{a.s.} \tag{3.25}$$

We now construct the desired Brownian motion. We first observe that for any B-valued Brownian motion $\{X(t),\ t \geqslant 0\}$ with covariance function given by (1.1) the sequences of d_k-dimensional random vectors

$$\left\{(t_k - t_{k-1})^{-1/2} P_{s_k}(X(t_k) - X(t_{k-1})),\ k \geqslant 1\right\} \quad \text{and} \quad \{Y_k,\ k \geqslant 1\}$$

have the same law since Y_k has Gaussian distribution G_k with the same covariance matrix as Z_k and thus as $P_{s_k} x_1$, by (3.23) and stationarity. For any integers l, m, n let $F = F(l, m)$ denote the joint distribution of $\{x_j,\ 1 \leqslant j \leqslant l\}$ and $\{Y_k,\ 1 \leqslant k \leqslant m\}$ and let $G = G(m, n)$ be the joint distribution of

$$\left\{(t_k - t_{k-1})^{-1/2} P_{s_k}(X(t_k) - X(t_{k-1})),\ 1 \leqslant k \leqslant m\right\} \quad \text{and} \quad \{X(r_i),\ 1 \leqslant i \leqslant n\}$$

where r_i are nonnegative numbers. Then by the above remarks the second marginal of F equals the first marginal of G. Hence by Lemma A1 of [**1**, p. 53], there is a probability space and three random elements ξ_i, $1 \leqslant i \leqslant 3$, such that ξ_1 has the same distribution as $\{x_j,\ 1 \leqslant j \leqslant l\}$, ξ_2 has the same distribution as $\{Y_k,\ 1 \leqslant k \leqslant m\}$ or as $\{(t_k - t_{k-1})^{-1/2} P_{s_k}(X(t_k) - X(t_{k-1})),\ 1 \leqslant k \leqslant m\}$ and ξ_3 has the same distribution as $\{X(r_i),\ 1 \leqslant i \leqslant n\}$. As l, m, n and $\{r_i,\ 1 \leqslant i \leqslant n\}$ vary this defines a consistent system H of distributions $H(l, m, n)$. Hence by Kolmogorov's theorem there exists a probability space on which we can redefine the sequences $\{x_j, j \geqslant 1\}$ and $\{Y_k,\ k \geqslant 1\}$ without changing their joint law and on which there exists a Brownian motion $\{X(t),\ t \geqslant 0\}$ with mean zero and covariance function given by (1.1) such that

$$(t_k - t_{k-1})^{-1/2} P_{s_k}(X(t_k) - X(t_{k-1})) = Y_k, \qquad k \geqslant 1. \tag{3.26}$$

We now show that $\{X(t),\ t \geqslant 0\}$ has the desired properties. Let $t > 0$ be given and define m by $t_{m-1} < t \leqslant t_m$. Then

$$\begin{aligned}
&\sup_{t_{m-1} < t \leqslant t_m} \Big\| \sum_{j \leqslant t} x_j - X(t) \Big\| \\
&\quad \leqslant \sup_{t_{m-1} < t \leqslant t_m} \Big\| \sum_{t_{m-1} < j \leqslant t} x_j \Big\| + \sup_{t_{m-1} < t \leqslant t_m} \| X(t) - X(t_{m-1}) \| \\
&\qquad + \sum_{1 \leqslant k < m} \Big\| \sum_{t_{k-1} < j \leqslant t_k} P_{s_k} x_j - P_{s_k}(X(t_k) - X(t_{k-1})) \Big\| \\
&\qquad + \sum_{1 \leqslant k < m} \Big\| \sum_{t_{k-1} < j \leqslant t_k} (x_j - P_{s_k} x_j) \Big\| \\
&\qquad + \sum_{1 \leqslant k < m} \| X(t_k) - X(t_{k-1}) - P_{s_k}(X(t_k) - X(t_{k-1})) \|,
\end{aligned} \tag{3.27}$$

where we take $t_0 = 0$. We shall consider these terms one by one. Call them I, II, III, IV and V. By Lemmas 3.4 and 3.5

$$\mathrm{I} + \mathrm{II} \ll u(s_m)^{\varepsilon/2} (t_m \log\log t_m)^{1/2} \quad \text{a.s.} \tag{3.28}$$

By (3.23), (3.25), (3.26), (3.5) and (3.10)

$$\mathrm{III} \ll \sum_{k < m} t_k^{1/2 - \lambda_1} \ll t_{m-1}^{1/2 - \lambda_1/2} \quad \text{a.s.} \tag{3.29}$$

By Lemmas 3.2, 3.3 and (3.8)

$$\text{(3.30)} \qquad \mathrm{IV} + \mathrm{V} \ll \sum_{k<m} u(s_k)^{3\varepsilon/2}(n_k \log\log t_k)^{1/2} \ll t_{m-1}^{1/2} u(s_{m-1})^{\varepsilon} \quad \text{a.s.}$$

It follows from (3.5) and (3.7) that $t_m \leqslant 2t_{m-1}$. Also from (1.6) and (3.14) that $u(s_{m-1}) \leqslant 2u(s_m)$. Thus we can extrapolate between t_{m-1} and t_m and s_{m-1} and s_m to get

$$\text{(3.31)} \qquad \left\| \sum_{j<t} x_j - X(t) \right\| \ll (u(\alpha/\log t))^{\varepsilon/2}(t \log\log t)^{1/2} \quad \text{a.s.}$$

Here we used (3.4). It is easy to see that the dominant term in (3.31) is contributed by (3.28). Note also that (1.6) implies that

$$\text{(3.32)} \qquad u^{\varepsilon}(\alpha/\log t) \geqslant \alpha/\log t$$

so that the bound in (3.29) is smaller than the bound in (3.28). The bound in (3.32) also justifies the argument following (3.16). This completes the proof of Case (i).

3.2. *Case* (ii). Here it is enough to simply choose $u_1(s_k) = s_k^{\beta}$. We define s_k and t_k as in (3.2) and (3.4) respectively. However, all the calculations are much simpler now because we get specific values for all quantities in terms of k. In particular

$$\text{(3.33)} \qquad s_k = (\alpha/k)^{1/(1+\beta)}.$$

We define n_k as in (3.5) and given (3.33) it is simple to obtain

$$\text{(3.34)} \qquad n_k \sim \alpha' t_k k^{-\beta/(1+\beta)} = \alpha'' t_k u_1(s_k)$$

for positive constants α', α'' depending on α and β. This gives us (3.7) for $n_k = n_{k,1}$. The critical inequality (3.8) now is an easy consequence of (3.7) using integration by parts, i.e.

$$\begin{aligned}\sum_{k<m} n_k^{1/2} u_1(s_k)^{3/2} \ll \sum_{k<m} t_k^{1/2} u_1^2(s_k) &\ll \sum_{k<m} \exp\left(\tfrac{1}{2}\alpha k^{1/(1+\beta)}\right) k^{-2\beta/(1+\beta)} \\ &\ll \exp\left(\tfrac{1}{2}\alpha m^{1/(1+\beta)}\right) m^{-\beta/(1+\beta)} \ll t_m^{1/2} u_1(s_m).\end{aligned}$$

Analogues of (3.9) and (3.10) follow immediately.

Now that we have the basic inequalities of Lemma 3.1 everything goes through as in Case (i) with the sole exception that we must take $\rho = 4((1+\beta)\log\log t_k)^{1/2}$ in Lemmas 3.2 and 3.3. We finally get (3.31) with $u(s) = s^{\beta}$. This completes the proof in Case (ii).

3.3. *Case* (iii). We define $s_k = k^{-1/\beta}$ and $t_k = s_k^{-\alpha} = k^{\alpha/\beta}$. It is completely elementary to obtain

$$\text{(3.35)} \qquad n_k \sim C_1 t_k k^{-1} = C_2 t_k u(s_k)$$

for some constants C_1 and C_2. Also

$$\text{(3.36)} \qquad \begin{aligned}\sum_{k<m} n_k u_1(s_k)^{3/2} \ll \sum_{k<m} t_k^{1/2} u_1^2(s_k) &\ll \sum_{k<m} k^{\alpha/(2\beta)-2} \\ &\ll m^{\alpha/(2\beta)-1} \ll t_m^{1/2} u_1(s_m).\end{aligned}$$

In place of inequality (3.9) we do the estimate of the Prohorov distance in (3.24) directly and get

$$\pi(F_k, G_k) \ll n_k^{-\delta/9} \cdot s_k^{-4/3} \ll k^{-3} = t_k^{-\lambda_2}, \qquad \lambda_2 > 0,$$

if $\alpha \geqslant (36\beta + 12)/\delta$.

As in Case (ii) we must redefine ρ in Lemmas 3.2 and 3.3. We choose $\rho = C_3(\log t_k)^{1/2} = C_4(\log k)^{1/2}$ where C_3 and C_4 are positive constants depending on α and β. It is easy to see that $\rho c \leqslant 1$. The term $\rho = C_4(\log k)^{1/2}$ can be absorbed into the $t_k^{-\lambda}$ term since $u(s_k) = k^{-1}$ and thus, by (3.35) n_k is of the right order. The same is true for the terms in (3.36).

4. Processes on $\mathbf{C}(K, \tau)$. Let (K, τ) be a compact metric or pseudometric space and, as usual, let $\mathbf{C}(K) = \mathbf{C}(K, \tau)$ be the Banach space of continuous complex-valued functions in the supremum norm $\|\cdot\|_\infty$. By treating real and imaginary parts as pairs we still can apply Theorem 1.1. We begin by relating the expression in (1.4) to a uniform Lipschitz norm on $\mathbf{C}(K)$. For $\varepsilon > 0$ let $N = N_\tau(s)$ denote the smallest number of open balls of radius s in the τ-metric or pseudometric, with centers in K, that covers K. Pick one such covering $\{U_i, i = 1, \ldots, N\}$ and let v_i denote the center of U_i. It is well known that there exists a partition of unity subordinated to $\{U_i\}$. That is, there exists a family of real valued functions $\{\phi_i, i = 1, \ldots, N\}$ defined on K such that $0 \leqslant \phi_i \leqslant 1$, $\sum_{i=1}^N \phi_i(y) = 1$ for all $y \in K$ and the support of each ϕ_i is contained in U_i. Following [**4**, §9] define, for $x \in \mathbf{C}(K)$,

$$P_s x = \sum_{i=1}^{N} x(v_i)\phi_i.$$

Clearly $\|P_s\| = 1$ so P_s is a bounded linear projection on $\mathbf{C}(K)$. The following lemma is taken from [**4**, §11].

LEMMA 4.1. *Let Z be a $\mathbf{C}(K, \tau)$-valued random variable and $T(s)$ an integer-valued function. $T(s) \uparrow \infty$ as $s \downarrow 0$. Suppose*

$$E\left[\sup_{\substack{\tau(v,v') \leqslant s \\ v,v' \in K}} |Z(v) - Z(v')|\right] = \psi(s). \tag{4.1}$$

Then there exists a projection P_s: $\mathbf{C}(K) \to \mathbf{C}(K)$ such that $\dim P_s = T(s)$ *and such that*

$$E\|Z - P_s Z\| \leqslant \psi\big(N_\tau^{-1}(T(s))\big) \tag{4.2}$$

where N_τ^{-1} is the inverse function of N_τ.

PROOF. We have

$$(x - P_s x)(v) = \sum_{i=1}^{N} (x(v) - x(v_i))\phi_i(v).$$

If $v \in K$, then $\phi_i(v) = 0$ unless $\tau(v, v_i) \leqslant s$. Therefore

$$\|x - P_s x\| \leqslant \sup_{\substack{\tau(v,v') \leqslant s \\ v,v' \in K}} |x(v) - x(v')|.$$

This together with (4.1) gives $E\|Z - P_s Z\| \leqslant \psi(s)$, where $\dim P_s = N_\tau(s)$. The inequality in (4.2) now results from a simple change of variables.

We shall now consider several processes for which we can obtain estimates of the function $u(s)$ in Theorem 1.1 and (1.4) via Lemma 4.1. Let $\{a_k, k \geqslant 1\} \in l^2$ be complex numbers, $\{\varepsilon_k, k \geqslant 1\}$ a Rademacher sequence (i.e. a sequence of independent random variables with $P(\varepsilon_k = 1) = P(\varepsilon_k = -1) = \frac{1}{2}$) and $\{\xi_k, k \geqslant 1\}$ a sequence of complex-valued random variables such that

$$\sup_k E^{1/2}|\xi_k|^2 \leqslant Q. \tag{4.3}$$

The random variables ξ_k need not be independent, but the sequences $\{\xi_k, k \geqslant 1\}$ and $\{\varepsilon_k, k \geqslant 1\}$ are assumed to be independent of each other.

(1) *Random Fourier series on a locally compact abelian group G.* Let Γ denote the character group of G and let K be a compact symmetric neighborhood of the identity of G. We will assume that Γ is countable and hence that K is metrizable. Let $\{\gamma_k, k \geqslant 1\}$ be some ordering of the elements of Γ and let τ be a metric on K. We consider the $C(K, \tau)$-valued random variables

$$x_1(v) = \sum_{k \geqslant 1} a_k \varepsilon_k \xi_k \gamma_k(v), \qquad v \in K, \tag{4.4}$$

and define a translation invariant pseudometric on G

$$\sigma_1(v, v') = \Big(\sum_{k \geqslant 1} |a_k|^2 |\gamma_k(v - v') - 1|^2 \Big)^{1/2}. \tag{4.5}$$

Note that if $v, v' \in K$ the domain of σ_1 is $K \oplus K$.

(2) *Uniformly Lipschitz continuous stochastic processes.* Let (K, τ) be a compact metric space and let $\{x_2(v), v \in K\}$ be a complex-valued stochastic process on the probability space (Ω, F, P) such that $Ex_2(v) =$ constant and such that $E|x(v_0)|^2 < \infty$ for some $v_0 \in K$. Let $\sigma_2(v, v')$ be a translation invariant pseudometric on K, continuous with respect to τ such that for all $v, v' \in K$,

$$|x_2(v) - x_2(v')| \leqslant M\sigma_2(v, v'), \tag{4.6}$$

where M is a random variable on (Ω, F, P) satisfying $EM^2 < \infty$.

The next two examples are complex-valued stochastic processes on $\mathbf{R}^N$. We continue to use the notation $\mathbf{C}(K, \tau)$, but now $K = [-\frac{1}{2}, \frac{1}{2}]^N$ and τ is the ordinary Euclidean metric. Let $\{\lambda_k, k \geqslant 1\}$ be a sequence of random variables with values in $\mathbf{R}^N$.

(3) *Symmetrized empirical characteristic processes.* Assume that the sequence $\{\xi_k, \lambda_k, k \geqslant 1\}$ is independent of $\{\varepsilon_k, k \geqslant 1\}$. The sequences $\{\xi_k, k \geqslant 1\}$ and $\{\lambda_k, k \geqslant 1\}$ need not be sequences of independent random variables nor do the sequences have to be independent of each other. Define

$$x_3(v) = \sum_{k \geqslant 1} a_k \varepsilon_k \xi_k \, e^{i\langle \lambda_k, v\rangle}, \qquad v \in K. \tag{4.7}$$

Let

$$\phi_k(v) = 2\big(E|\xi_k|^2 \sin^2 \tfrac{1}{2}\langle \lambda_k, v\rangle\big)^{1/2}, \qquad v \in [-1, 1]^N, \tag{4.8}$$

and define the translation-invariant metric or pseudometric

$$\sigma_3(v, v') = \Big(\sum_{k \geq 1} |a_k|^2 \phi_k^2(v - v') \Big)^{1/2}. \tag{4.9}$$

(4) *Empirical characteristic process.* We now assume that $\{(\xi_k, \lambda_k), k \geq 1\}$ is a sequence of independent random variables ($\{\xi_k, k \geq 1\}$ and $\{\lambda_k, k \geq 1\}$ need not be independent of each other). Let

$$c_k(v) = E\xi_k e^{i\langle \lambda_k, v\rangle}, \qquad v \in K. \tag{4.10}$$

We define

$$x_4(v) = \sum_{k \geq 1} a_k\big(\xi_k e^{i\langle \lambda_k, v\rangle} - c_k(v)\big), \qquad v \in K, \tag{4.11}$$

and σ_4 the same way as σ_3 given in (4.9).

Let $\{x_{ij}, j \geq 1\}$ be a sequence of independent copies of x_i, $i \leq i \leq 4$, and define

$$S_{in} = \sum_{j < n} x_{ij}. \tag{4.12}$$

Furthermore, for all these processes we assume

$$\int_0^\infty \big(\log N_{\sigma_i}(K, \varepsilon)\big)^{1/2} \, d\varepsilon < \infty, \qquad 1 \leq i < 4. \tag{4.13}$$

THEOREM 4.2. *Let x_i, $1 \leq i \leq 4$, be the stochastic processes defined above. Assume that* (4.13) *holds and let S_{in} be as defined in* (4.12). *Then for* $1 \leq i \leq 4$,

$$\begin{aligned} &\sup_{n \geq 1} n^{-1/2} E\Big\{ \sup_{\tau(v,v') < s} |S_{in}(v) - S_{in}(v')| \Big\} \\ &\qquad \leq C\Big(\int_0^{\hat\sigma_i(s)} \big(\log N_{\sigma_i}(K, \varepsilon)\big)^{1/2} \, d\varepsilon + \hat\sigma_i(s) \Big) = \psi_i(s), \end{aligned} \tag{4.14}$$

where C is a finite positive constant and

$$\hat\sigma_1(s) = E^{1/2}\Big\{ \sup_{\tau(v,v') \leq s} \sum_{k \geq 1} |a_k|^2 |\xi_k|^2 |\gamma_k(v - v') - 1|^2 \Big\}, \tag{4.15}$$

$$\hat\sigma_2(s) = \sup_{\tau(v,v') \leq s} \sigma_2(v, v'), \tag{4.16}$$

and, for $i = 3, 4$,

$$\hat\sigma_i(s) = 2E^{1/2}\Big\{ \sup_{\tau(v,v') < s} \sum_{k \geq 1} |a_k|^2 |\xi_k|^2 \sin^2\langle \lambda_k, v - v'\rangle \Big\}. \tag{4.17}$$

Furthermore, $\lim_{s \to 0} \hat\sigma_i(s) = 0$, $1 \leq i \leq 4$, *and this together with* (4.14) *gives* $\lim_{s \to 0} \psi_i(s) = 0$, $1 \leq i \leq 4$.

PROOF. The proof for $i = 1$ follows immediately from Theorem 1.4, Chapter 3, of **[16]** as explained in the proof of Theorem 1.1, Chapter 4, of **[16]**.

The proof for $i = 3$ is the same as that for $i = 1$ except that the expectation (see (1.23), Chapter 3, of **[16]**) must also be taken with respect to $\{\lambda_k\}$. We will outline the beginning of the proof. Let $\{\varepsilon_k\}$ be defined on the probability space (Ω_2, F_2, P_2) and $\{\xi_k, \lambda_k\}$ on (Ω_1, F_1, P_1) and let E_2 and E_1 be the corresponding expectation operators. Let (Ω, F, P) denote the product probability space $(\Omega_1 \times \Omega_2, F_1 \times F_2, P_1 \times P_2)$ and denote the expectation operator on this space by E. It is obvious that

$$\sigma_3(v - v') = n^{-1/2} E^{1/2} |S_{3n}(v) - S_{3n}(v')|^2.$$

Consider, for fixed $\omega_1 \in \Omega_1$

$$n^{-1/2} S_{3n}(v, \omega_1) = \sum_{k \geq 1} a_k n^{-1/2} \sum_{j \leq n} \varepsilon_{kj} \xi_{kj}(\omega_1) e^{i\langle \lambda_{kj}(\omega_1), v\rangle}$$

and

$$\begin{aligned}\sigma_{3n}(v - v', \omega_1) &= n^{-1/2} E_2^{1/2} |S_{3n}(v, \omega_1) - S_{3n}(v', \omega_1)|^2 \\ &= 2\left(\sum_{k \geq 1} |a_k|^2 |\xi_k(\omega_1)|^2 \sin^2 \tfrac{1}{2} \langle \lambda_k(\omega_1), v - v' \rangle \right)^{1/2}.\end{aligned}$$

Note that $\sigma_{3n} = \sigma_{31}$ for all n. Let

$$\hat{\sigma}_3(s, \omega_1) = \sup_{\tau(v,v') \leq s} \sigma_{3n}(v - v', \omega_1) \quad \text{and} \quad \hat{\sigma}_3(s) = E_1^{1/2} (\hat{\sigma}_3(s, \omega_1))^2.$$

Following the proof of Theorem 1.4, Chapter 3, of **[16]** we have

$$\begin{aligned}&E_2 \sup_{\tau(v,v') \leq s} |S_{3n}(v, \omega_1) - S_{3n}(v', \omega_1)| \\ &\qquad \leq E_2 \sup_{\sigma_{3n}(v - v', \omega_1) \leq \hat{\sigma}_{3n}(s, \omega_1)} |S_{3n}(v, \omega_1) - S_{3n}(v', \omega_1)|.\end{aligned}$$

Note that $S_{3n}(v, \omega_1)$ is a subgaussian process. Therefore we can continue the proof with the third inequality in (1.21), Chapter 3, of **[16]**, with the minor exception that $g(a)$ in that paper is now defined as $E_1^{1/2} (\hat{\sigma}_3(a, \omega_1))^2$ instead of $E_1 \hat{\sigma}_3(a, \omega_1)$.

The case $i = 4$ follows immediately from the case $i = 3$ since by (10) of **[15]**

$$E \sup_{\tau(v,v') \leq s} |S_{4n}(v) - S_{4n}(v')| \leq 2E \sup_{\tau(v,v') \leq s} |S_{3n}(v) - S_{3n}(v')|.$$

Lastly we consider the case $i = 2$. The proof here is much easier than in the other cases. It follows from Theorem 3.1, Chapter 2, of **[16]**, which is a generalization of a familiar theorem of Dudley. The idea, in the symmetric case, is to consider

$$\tilde{S}_n(v, \omega_1) = \sum_{j \leq n} \varepsilon_k x_{2j}(v, \omega_1) / \sum_{j \leq n} M_j^2(\omega_1)$$

as a subgaussian process. This satisfies

$$E_2^{1/2} |\tilde{S}_n(v, \omega_1) - \tilde{S}_n(v', \omega_1)|^2 \leq \sigma_2(v, v').$$

Also we have

$$\begin{aligned}&E_2 \sup_{\tau(v,v') \leq s} |\tilde{S}_n(v, \omega_1) - \tilde{S}_n(v', \omega_1)| \\ &\qquad \leq E \sup_{\sigma_2(v,v') \leq \hat{\sigma}_2(s)} |\tilde{S}_n(v, \omega_1) - \tilde{S}_n(v', \omega_1)|.\end{aligned} \tag{4.18}$$

The desired result, (4.14), now follows by applying Theorem 3.1, Chapter 2, of **[16]**, to (4.18) and then applying the expectation operator E_1. To desymmetrize we use Lemma 2 of **[15]**.

Finally the fact that $\lim_{s\to 0} \hat{\sigma}_i(s) = 0$ follows from the dominated convergence theorem for $i = 1, 3, 4$ and by hypothesis in the case $i = 2$. □

REMARK 4.3. It is easy to see that $\sup_{v\in K} E|x_i(v)|^2 < \infty$, $1 \leqslant i \leqslant 4$. This imples that for any $v_1, \ldots, v_m \in K$ the vector $(x_i(v_1), \ldots, x_i(v_m))$ satisfies the central limit theorem on $\mathbf{R}^m$, $1 \leqslant i \leqslant 4$. This together with Theorem 4.2 shows that the processes x_i, $1 \leqslant i \leqslant 4$, satisfy the central limit theorem on $\mathbf{C}(K, \tau)$ for the appropriate space (K, τ). For the processes in (1) this result was obtained in **[14]** and extended in **[16]**, (see Theorem 1.1, Chapter 4, of **[16]**) and for the processes in (2) it was obtained in **[8]**. For certain processes in (3) and (4) it was obtained in **[15]** and, in general, as we saw above, the central limit theorem for the processes in (3) and (4) is an immediate corollary of Theorem 1.4, Chapter 3, of **[16]**. It is interesting to note that in the cases $i = 1, 3, 4$, the limiting Gaussian measure exists only if certain stationary Gaussian processes exist and these exist if and only if (4.13) holds. Thus, for the processes of examples (1), (3) and (4) condition (4.13) is necessary and sufficient for the central limit theorem.

Lemma 4.1 and Theorem 4.2 enable us to apply Theorem 1.1 to the four classes of stochastic processes considered here. In the remainder of this section we will introduce various simplifying assumptions that enable us to make more explicit computations and we will give a proof of Theorem 1.2.

In all that follows in this section we will restrict our attention to the case $K = [-\frac{1}{2}, \frac{1}{2}]^N$ with metric $\tau(v, v') = |v - v'|$. We will obtain an upper bound for the integral in (4.12) when σ_i is translation invariant. Of course, for the processes in (1), (3) and (4) σ_i is translation invariant and for the processes in (2) this is an extra hypothesis.

Now let $\sigma(v, v')$ be a translation invariant metric on K. For notational convenience we write $\sigma(v, v') = \sigma(0, v - v') = \sigma(v - v')$. We define

$$m_\sigma(y) = \lambda\left\{z \in [-1, 1]^N : \sigma(z) < y\right\} \tag{4.19}$$

where λ is Lebesgue measure.

LEMMA 4.4. *Let $K = [-\frac{1}{2}, \frac{1}{2}]^N$ and let σ be a translation invariant metric on K. Write*

$$\hat{\sigma}(s) = \sup_{|u| < s} \sigma(u). \tag{4.20}$$

Then for $0 < s \leqslant \frac{1}{2}$ we have

$$\int_0^{\hat{\sigma}(s)} (\log N_\sigma(K, \varepsilon))^{1/2}\, d\varepsilon \leqslant C\left[\hat{\sigma}(s)(\log 1/s)^{1/2} + \int_0^s \frac{\hat{\sigma}(u)}{u(\log 1/u)^{1/2}}\, du\right] \tag{4.21}$$

where C is a constant depending on the dimension N.

PROOF. By (1.2) and (1.9), Chapter 2, of **[16]** we have

$$N_\sigma(K, \varepsilon) \leqslant 4^N / m_\sigma\left(\tfrac{1}{4}\varepsilon\right).$$

Therefore,

$$\int_0^{\hat\sigma(s)} (\log N_\sigma(K,\varepsilon))^{1/2}\,d\varepsilon \leqslant 4\int_0^s \left(\log \frac{4^N}{m_\sigma(\hat\sigma(v))}\right)^{1/2} d\hat\sigma(v). \tag{4.22}$$

By (4.19) and (4.20) $m_\sigma(\hat\sigma(v)) \geqslant \lambda\{u \in [-1,1]^N\colon |u| \leqslant v\} = B_N v^N$ for $0 \leqslant v \leqslant \frac{1}{2}$, where B_N is a constant depending on N. Using this in (4.22) we obtain (4.21).

REMARK 4.5. In order to use this lemma on the four types of processes considered we need to check that $\sup_{|u|\leqslant s} \sigma_i(u) = \hat\sigma_i(s)$. For $i = 4$ this follows from (4.8), (4.9) and (4.17). The case $i = 3$ is identical, the case $i = 2$ is trivial and the case $i = 1$ is equally simple.

LEMMA 4.6. *Let $K = [-\frac{1}{2}, \frac{1}{2}]^N$. Consider the processes x_3 and x_4 with the further assumption that the sequences $\{\xi_k\}$ and $\{\lambda_k\}$ are independent of each other. Let*

$$R_k(x) = P(|\lambda_k| \leqslant x). \tag{4.23}$$

Then for $i = 3$ and 4,

$$\hat\sigma_i(s) \leqslant Q\Big(\sum_{k\geqslant 1} |a_k|^2 G_k(s)\Big)^{1/2},$$

where Q is given in (4.3) *and*

$$G_k(s) = 2s^2\int_0^{1/s} x(1 - R_k(x))\,dx. \tag{4.24}$$

PROOF. By (4.17) for $i = 3, 4$,

$$\begin{aligned}\hat\sigma_i^2(s) &= 4E\sup_{|u|<s}\sum_{k\geqslant 1} |a_k|^2|\xi_k|^2 \sin^2 \tfrac{1}{2}\langle \lambda_k, u\rangle \\ &\leqslant E\sum_{k\geqslant 1} |a_k|^2|\xi_k|^2\big(|\lambda_k|^2 s^2 \wedge 1\big) \leqslant Q\sum_{k\geqslant 1} |a_k|^2 E\big(|\lambda_k|^2 s^2 \wedge 1\big).\end{aligned} \tag{4.25}$$

But by a simple integration by parts

$$E\big(|\lambda_k|^2 s^2 \wedge 1\big) = s^2\int_0^{1/s} x^2 dR_k(x) - (1 - R_k(1/s)) = G_k(x).$$

COROLLARY 4.7. *In particular, if in the case $i = 3$ the random vectors λ_k are constants*

$$x_3(v) = x_1(v) = \sum_{k\geqslant 1} a_k\varepsilon_k\xi_k e^{i\langle\lambda_k, v\rangle}, \qquad v \in K, \tag{4.26}$$

is a random Fourier series and we have

$$\hat\sigma_3(s) = \hat\sigma_1(s) \leqslant Q\Big(2s^2\int_0^{1/s} v(1 - T(v))\,dv\Big)^{1/2} \tag{4.27}$$

where

$$T(v) = \sum_{|\lambda_k|<v} |a_k|^2. \tag{4.28}$$

PROOF. This can be obtained directly from Lemma 4.6, but easier still from the last line of (4.25) since

$$\sum_{k>1} |a_k|^2(|\lambda_k|^2 s^2 \wedge 1) = s^2 \sum_{|\lambda_k|s \leqslant 1} |a_k|^2|\lambda_k|^2 + \sum_{|\lambda_k|s>1} |a_k|^2$$
$$= s^2\int_0^{1/s} v^2 dT(v) - (1 - T(1/s)) = 2s^2\int_0^{1/s} v(1 - T(v))\, dv.$$

PROOF OF THEOREM 1.2. We apply the above results to the random variable $y(v) = \exp(i\langle X, v\rangle) - c(v)$, $v \in K$, where X is a random variable with values in $\mathbf{R}^N$ and $c(v)$ is its characteristic function. This is a special case of the processes given in (4). Since y is a bounded random variable we will take $\varepsilon = \frac{1}{4}$ in (1.5). Recall that (K, τ) is $[-\frac{1}{2}, \frac{1}{2}]^N$ with the Euclidean metric. Thus we have $N_\tau(s) > (B_N/s)^N$ for some constant B_N depending only upon N. Therefore $N_\tau^{-1}(s) \leqslant B_N s^{-1/N}$. In Theorem 1.1(i) dim $P_s = \exp(1/s)$. Therefore by Lemma 4.1 and Theorem 4.2 with $i = 4$ we obtain for all $n \geqslant 1$,

$$n^{-1/2}E\|S_n - P_s S_n\| \leqslant \psi_4(B_N \exp(-1/(Ns))). \tag{4.29}$$

Let w be a concave majorant of $\psi_4^{1/4}$ such that $\lim_{s\to 0} w(s) = 0$. Such a function w exists since by Theorem 4.2 $\lim_{s\to 0}\psi_4(s) = 0$. Then by (1.7) and (1.8) and since $\varepsilon = \frac{1}{4}$ is an admissible choice we obtain (1.22). (The functions ψ in (1.21) and ψ_4 in (4.14) are the same.)

To prove the remainder of Theorem 1.2 we first observe that by (4) the metric associated with the process y, as defined above, is translation invariant and given by

$$\sigma_4(v) = \sigma_4(0, v) = 2\left(E \sin^2\tfrac{1}{2}\langle X, u\rangle\right)^{1/2}$$

and that by Lemma 4.6

$$\hat{\sigma}_4(s) = \left(2s^2\int_0^{1/s} x(1 - R(x))\, dx\right)^{1/2}$$

where $R(x) = P(|X| \leqslant x)$. Hence (1.23) implies

$$\hat{\sigma}_4(s) \ll (\log 1/s)^{-1/2}\left(\log\log \frac{1}{s}\right)^{-g}, \qquad s \to 0,$$

and thus by (4.14) and (4.21)

$$\psi_4(s) \ll (\log\log 1/s)^{-g+1}. \tag{4.30}$$

Similarly (1.25) implies

$$\psi_4(s) \ll (\log 1/s)^{-(g-1)/2} \tag{4.31}$$

and (1.27) implies

$$\psi_4(s) \ll s^{g/2}(\log 1/s)^{1/2}. \tag{4.32}$$

Hence if (1.23) holds, (1.24) follows directly from (1.8), (4.29) and (4.30). In the same way (1.25) implies (1.26). Finally, if (1.27) holds we have dim $P(s) = 1/s$ and by Lemma 4.1,

$$n^{-1/2}E\|S_n - P_s S_n\| \leqslant \psi_4\left(B_N s^{1/N}\right). \tag{4.33}$$

(1.28) follows now from (1.10) and (4.33).

REMARK 4.8. From the proof of Theorem 1.2 it is clear that it also holds for the sequence $\{\varepsilon_j \exp(i\langle X_j, v\rangle), j \geq 1, v \in [-\frac{1}{2}, \frac{1}{2}]^N\}$ where X_j are independent, identically distributed copies of the $\mathbf{R}^N$-valued random variable X. This is because the functions R and σ are the same for this process. In this case the approximating Gaussian process is $\{H_1(v, t), v \in [-\frac{1}{2}, \frac{1}{2}]^N, t \geq 0\}$ with mean zero and covariance function

$$E\left\{H_1(v, t)\overline{H_1(v', t')}\right\} = \min(t, t')c(v - v').$$

Similarly, Theorem 1.2 remains valid for the partial sum process of independent copies of the process considered in (4.26) if $R(x)$ in Theorem 1.2 is replaced by $T(x)$ given in (4.28). In this case the approximating Gaussian process is $\{H_2(v, t), v \in [-\frac{1}{2}, \frac{1}{2}]^N, t \geq 0\}$ with mean zero and covariance function

$$E\left\{H_2(v, t)\overline{H_2(v', t')}\right\} = \min(t, t') \sum_{k \geq 1} |a_k|^2 E|\xi_k|^2 e^{i\langle\lambda_k, v - v'\rangle}.$$

Also in this case we need $E\|x_1\|^{2+\delta} < \infty$. (See the next remark.)

REMARK 4.9. Theorem 1.1 requires that $E\|x_1\|^{2+\delta} < \infty$. We shall consider this condition for the four examples of processes given at the beginning of this section. For the processes in (2) it is easy to see that $E|x_2(v_0)|^{2+\delta} < \infty$ for some $v_0 \in K$ and $E|M|^{2+\delta} < \infty$ is sufficient. For the processes in (1), (3) and (4) assume that $\{(\xi_k, \lambda_k), k \geq 1\}$ is a sequence of independent random variables. Then by Theorem 3.3 of **[9]** and (4.13) we have $E\|x_i\|^{2+\delta} < \infty$ if and only if $\sup_{k \geq 1} E|\xi_k|^{2+\delta} < \infty$, $i = 1, 3, 4$.

In the general case of the processes given in (1) it is again true that if (4.13) holds then $E\|x_1\|^{2+\delta} < \infty$ if and only if $\sup_{k \geq 1} E|\xi_k|^{2+\delta} < \infty$. This is proved by following the proof of Theorem 1.1, Chapter 3, of **[16]** but with

$$\|\cdot\|_\Omega = \left(E|\cdot|^{2+\delta}\right)^{1/(2+\delta)}$$

in (1.7). Also Corollary 4.6, Chapter 2, of **[16]** must be used in (1.3) of **[16]**. Finally, this same method can be used for the processes in (3) when $\{(\xi_k, \lambda_k), k \geq 1\}$ is not a sequence of independent random variables. However, our approach requires additional hypotheses besides (4.13) and $\sup_{k \geq 1} E|\xi_k|^{2+\delta} < \infty$. We will not go into the details here.

REMARK 4.10. These methods can also be used on the second order stochastic integrals considered by Fernique **[5]**. Theorems 4.2 and 1.2 can also be given for these processes.

5. Further applications. Let $\{x_j, j \geq 1\}$ be a sequence of independent identically distributed B-valued random variables, with mean zero, finite second moment and satisfying the central limit theorem. Then, according to a theorem of Pisier **[19]**, $\{x_j, j \geq 1\}$ also satisfies the compact law of the iterated logarithm with cluster set determined by the covariance function $E\{f(x_1)g(x_1)\}$, $f, g \in B^*$. But this also follows from (1.2) and (1.1) since, as is well known Brownian motion satisfies the compact law of the iterated logarithm with cluster set determined by $E\{f(X(1))g(X(1))\}, f, g \in B^*$.

Upper and lower class refinements of the law of the iterated logarithm can easily be obtained from corresponding results for B-valued Brownian motion provided that in relation (1.7) the function g satisfies

$$g(t, \delta) \ll 1/\log\log t. \tag{5.1}$$

Indeed, consider Theorem 2.3 of Kuelbs **[11]** which is such an upper and lower class result for Brownian motion. It involves the integral

$$\int^{\infty} \frac{(\phi(t))^{n_1}}{t} e^{-\phi^2(t)/2}\, dt. \tag{5.2}$$

where n_1 is some finite integer. As noted in (2.5) of **[11]** in the proof of Theorem 2.3 of **[11]** there is no loss of generality to assume $(\log\log t)^{1/2} \leqslant \phi(t) \leqslant 2(\log\log t)^{1/2}$ for all sufficiently large t. Hence the convergence or divergence of the integral (5.2) is not affected if a function of size $\ll (\log\log t)^{-1/2}$ is added to $\phi(t)$. But if (5.1) holds then (1.7) becomes

$$\Big\| \sum_{j<t} x_j - X(t) \Big\| \ll (t/\log\log t)^{1/2} \quad \text{a.s.}$$

Consequently, Theorem 2.3 of Kuelbs **[11]** continues to hold for the partial sum process.

Another application deals with the simulation of the sample paths of Gaussian processes. Let $Y(v)$, $v \in [-\frac{1}{2}, \frac{1}{2}]^N$, be a Gaussian process with spectral representation $\int_{\mathbf{R}^N} e^{i\langle v,\lambda\rangle}\, db(F(\lambda))$ where b is Brownian motion and F is a distribution function on $\mathbf{R}^N$ called the spectrum of Y. Let $\{Y_j, j \geqslant 1\}$ be a sequence of independent copies of Y. Let $\{X_j, j \geqslant 1\}$ be a sequence of independent identically distributed $\mathbf{R}^N$-valued random variables with distribution function F. It follows from Theorem 1.1 and Remark 4.8 that with probability 1

$$\sup_{v\in K} \Big| n^{-1/2} \sum_{j<n} \varepsilon_j e^{i\langle X_j, v\rangle} - n^{-1/2} \sum_{j<n} Y_j \Big| \ll g(n, \delta)(\log\log n)^{1/2} \tag{5.3}$$

where $K = [-\frac{1}{2}, \frac{1}{2}]^N$. According to conditions on $R(x) = P\{|X| \leqslant x\}$ the term on the right in (5.3) will go to zero at various rates. Since $n^{-1/2}\sum_{j<n} Y_j$ is a process equal in law to Y this suggests the possibility of simulating a path of $Y(v)$ by

$$n^{-1/2} \sum_{j<n} \varepsilon_j e^{i\langle X_j, v\rangle}, \qquad v \in K. \tag{5.4}$$

The point is that it is probably easier to "generate" the sequence $\{X_j, j \geqslant 1\}$ with distribution function F and to form (5.4) than to create sample paths of a Gaussian process with spectrum F. The term on the right in (5.3) tells how good the approximation is. Actually, what is perhaps even more desirable is an estimate of the form

$$P\Big\{ \sup_{v\in K} \Big| n^{-1/2} \sum_{j<n} e^{i\langle X_j, v\rangle} - Y(v) \Big| \geqslant \varepsilon \Big\} \leqslant h(n, \varepsilon)$$

where $h(n, \varepsilon) \to 0$ as $n \to \infty$ for each $\varepsilon > 0$. This can easily be obtained from our methods, by using the tail estimates we give before we apply the Borel Cantelli lemmas.

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